STAR Research Journal

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MATHEMATICS

ON THE POSITIVE PELL EQUATION $y^2 = 112x^2 + 9$

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Abstract

The binary quadratic equation represented by the positive pellian $y^2 = 112x^2 + 9$ *is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special Pythagorean triangle.*

Keywords: Binary quadratic, hyperbola, parabola, integral solutions, pell equation.
2010 Mathematics Subject Classification: 11D09.

INTRODUCTION

The binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is a non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values[1-4]. For an extensive review of various problems, one may refer [5-20]. In this communication, yet another interesting hyperbola given by $y^2 = 112x^2 + 9$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

METHOD OF ANALYSIS

The positive pell equation representing hyperbola under consideration is

$$y^2 = 112x^2 + 9 \qquad (1)$$

whose smallest positive integer solution is $x_0 = 1, y_0 = 11$.

To obtain the other solutions of (1), consider the pell equation $y^2 = 112x^2 + 1$ whose solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{112}} g_n \ ,$$

$$\tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(127 + 12\sqrt{112}\right)^{n+1} + \left(127 - 12\sqrt{112}\right)^{n+1},$$

$$g_n = \left(127 + 12\sqrt{112}\right)^{n+1} - \left(127 - 12\sqrt{112}\right)^{n+1}$$

Applying Brahamagupta Lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$2\sqrt{112}x_{n+1} = \sqrt{112} f_n + 11 g_n ,$$

$$2y_{n+1} = 11 f_n + \sqrt{112} g_n \qquad \text{where } n = 0,1,2,\ldots..$$

The recurrence relations satisfied by the solutions x and y are given by

$$x_{n+1} - 254x_{n+2} + x_{n+3} = 0,$$
$$y_{n+1} - 254y_{n+2} + y_{n+3} = 0.$$

Some numerical examples of x and y satisfying (1) are given in the Table1 below.

Table1: Examples

n	x_n	y_n
0	1	11
1	259	2741
2	65785	696203
3	16709131	176832821
4	4244053489	44914840331

From the above table, we observe some interesting relations among the solutions which are presented below.

1) Both the values of x_n and y_n are odd.
2) Each of the following expressions is a nasty number.

- $\dfrac{11x_{2n+3} - 2741x_{2n+2} + 108}{9}$
- $\dfrac{11x_{2n+4} - 696203x_{2n+2} + 27432}{2286}$
- $\dfrac{44y_{2n+3} - 116032x_{2n+2} + 4572}{381}$
- $\dfrac{44y_{2n+4} - 29471680x_{2n+2} + 1161252}{96771}$
- $\dfrac{2741x_{2n+4} - 696203x_{2n+3} + 108}{9}$
- $\dfrac{10964y_{2n+2} - 448x_{2n+3} + 4572}{381}$
- $\dfrac{10964y_{2n+3} - 116032x_{2n+3} + 36}{3}$
- $\dfrac{10964y_{2n+4} - 29471680x_{2n+3} + 4572}{381}$
- $\dfrac{2784812y_{2n+2} - 448x_{2n+4} + 1161252}{96771}$
- $\dfrac{2784812y_{2n+3} - 116032x_{2n+4} + 4572}{381}$
- $\dfrac{2784812y_{2n+4} - 29471680x_{2n+4} + 36}{3}$
- $\dfrac{259y_{2n+2} - y_{2n+3} + 108}{9}$

- $\frac{65785y_{2n+2} - y_{2n+4} + 27432}{2286}$
- $\frac{65785y_{2n+3} - 259y_{2n+4} + 108}{9}$

3) Each of the following expressions is a cubical integer.

- $\frac{11x_{3n+4} - 2741x_{3n+3} + 33x_{n+2} - 8223x_{n+1}}{54}$
- $\frac{11x_{3n+5} - 696203x_{3n+3} + 33x_{n+3} - 2088609x_{n+1}}{13716}$
- $\frac{22y_{3n+4} - 58016x_{3n+3} + 66y_{n+2} - 174048x_{n+1}}{1143}$
- $\frac{22y_{3n+5} - 14735840x_{3n+3} + 66y_{n+3} - 44207520x_{n+1}}{290313}$
- $\frac{2741x_{3n+5} - 696203x_{3n+4} + 8223x_{n+3} - 2088609x_{n+2}}{54}$
- $\frac{5482y_{3n+3} - 224x_{3n+4} + 16446y_{n+1} - 672x_{n+2}}{1143}$
- $\frac{5482y_{3n+4} - 58016x_{3n+4} + 16446y_{n+2} - 174048x_{n+2}}{9}$
- $\frac{5482y_{3n+5} - 14735840x_{3n+4} + 16446y_{n+3} - 44207520x_{n+2}}{1143}$
- $\frac{1392406y_{3n+3} - 224x_{3n+5} + 4177218y_{n+1} - 672x_{n+3}}{290313}$
- $\frac{1392406y_{3n+4} - 58016x_{3n+5} + 4177218y_{n+2} - 174048x_{n+3}}{1143}$
- $\frac{1392406y_{3n+5} - 14735840x_{3n+5} + 4177218y_{n+3} - 44207520x_{n+3}}{9}$
- $\frac{259y_{3n+3} - y_{3n+4} + 777y_{n+1} - 3y_{n+2}}{54}$
- $\frac{65785y_{3n+3} - y_{3n+5} + 197355y_{n+1} - 3y_{n+3}}{13716}$
- $\frac{65785y_{3n+4} - 259y_{3n+5} + 197355y_{n+2} - 777y_{n+3}}{54}$

4) Relations among the solutions.

- $x_{n+3} = 254x_{n+2} - x_{n+1}$
- $12y_{n+1} = x_{n+2} - 127x_{n+1}$
- $12y_{n+2} = 127x_{n+2} - x_{n+1}$
- $3048y_{n+1} = x_{n+3} - 32257x_{n+1}$
- $24y_{n+2} = x_{n+3} - x_{n+1}$
- $3048y_{n+3} = 32257x_{n+3} - x_{n+1}$

- $127y_{n+1} = y_{n+2} - 1344x_{n+1}$
- $32257y_{n+1} = y_{n+3} - 341376x_{n+1}$
- $32257y_{n+2} = 127y_{n+3} - 1344x_{n+1}$
- $12y_{n+1} = 127x_{n+3} - 32257x_{n+2}$
- $12y_{n+2} = x_{n+3} - 127x_{n+2}$
- $12y_{n+3} = 127x_{n+3} - x_{n+2}$
- $127y_{n+2} = y_{n+1} + 1344x_{n+2}$
- $y_{n+3} = y_{n+1} - 2688x_{n+2}$
- $y_{n+3} = 127y_{n+2} + 1344x_{n+2}$
- $127x_{n+1} = 32257x_{n+2} - 12y_{n+3}$
- $32257y_{n+2} = 127y_{n+1} + 1344x_{n+3}$
- $32257y_{n+3} = 341376x_{n+3} + y_{n+1}$
- $127y_{n+3} = 1344x_{n+3} + y_{n+2}$
- $y_{n+3} = 254y_{n+2} - y_{n+1}$

REMARKABLE OBSERVATIONS

I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the Table2 below.

Tabsle2: Hyperbolas

S. No	(X,Y)	Hyperbola
1	$\left(259\sqrt{112}x_{n+1} - \sqrt{112}x_{n+2}, 11x_{n+2} - 2741x_{n+1}\right)$	$Y^2 - X^2 = 11664$
2	$\left(65785\sqrt{112}x_{n+1} - \sqrt{112}x_{n+3}, 11x_{n+3} - 696203x_{n+1}\right)$	$Y^2 - X^2 = 752514624$
3	$\left(5482\sqrt{112}x_{n+1} - 2\sqrt{112}y_{n+2}, 22y_{n+2} - 58016x_{n+1}\right)$	$Y^2 - X^2 = 5225796$
4	$\left(1392406\sqrt{112}x_{n+1} - 2\sqrt{112}y_{n+3}, 22y_{n+3} - 14735840x_{n+1}\right)$	$Y^2 - X^2 = 337126551876$
5	$\left(65785\sqrt{112}x_{n+2} - 259\sqrt{112}x_{n+3}, 2741x_{n+3} - 696203x_{n+2}\right)$	$Y^2 - X^2 = 11664$
6	$\left(22\sqrt{112}x_{n+2} - 518\sqrt{112}y_{n+1}, 5482y_{n+1} - 224x_{n+2}\right)$	$Y^2 - X^2 = 5225796$
7	$\left(5482\sqrt{112}x_{n+2} - 518\sqrt{112}y_{n+2}, 5482y_{n+2} - 58016x_{n+2}\right)$	$Y^2 - X^2 = 324$
8	$\left(1392406\sqrt{112}x_{n+2} - 518\sqrt{112}y_{n+3}, 5482y_{n+3} - 14735840x_{n+2}\right)$	$Y^2 - X^2 = 5225796$
9	$\left(22\sqrt{112}x_{n+3} - 131570\sqrt{112}y_{n+1}, 1392406y_{n+1} - 224x_{n+3}\right)$	$Y^2 - X^2 = 337126551876$

10	$\left(5482\sqrt{112}x_{n+3} - 131570\sqrt{112}y_{n+2}, 1392406y_{n+2} - 58016x_{n+3}\right)$	$Y^2 - X^2 = 5225796$
11	$\left(1392406\sqrt{112}x_{n+3} - 131570\sqrt{112}y_{n+3}, 1392406y_{n+3} - 14735840x_{n+3}\right)$	$Y^2 - X^2 = 324$
12	$\left(11y_{n+2} - 2741y_{n+1}, 259y_{n+1} - y_{n+2}\right)$	$112Y^2 - X^2 = 1306368$
13	$\left(11y_{n+3} - 696203y_{n+1}, 65785y_{n+1} - y_{n+3}\right)$	$112Y^2 - X^2 = 84281637888$
14	$\left(2741y_{n+3} - 696203y_{n+2}, 65785y_{n+2} - 259y_{n+3}\right)$	$112Y^2 - X^2 = 1306368$

II. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table3 below.

Table3: Parabolas

S. No	(X,Y)	parabola
1	$\left(259\sqrt{112}x_{n+1} - \sqrt{112}x_{n+2}, 11x_{2n+3} - 2741x_{2n+2}\right)$	$X^2 = 54Y - 11664$
2	$\left(65785\sqrt{112}x_{n+1} - \sqrt{112}x_{n+3}, 11x_{2n+4} - 696203x_{2n+2}\right)$	$X^2 = 13716Y - 752514624$
3	$\left(5482\sqrt{112}x_{n+1} - 2\sqrt{112}y_{n+2}, 22y_{2n+3} - 58016x_{2n+2}\right)$	$X^2 = 1143Y - 5225796$
4	$\left(1392406\sqrt{112}x_{n+1} - 2\sqrt{112}y_{n+3}, 22y_{2n+4} - 14735840x_{2n+2}\right)$	$X^2 = 290313Y - 337126551876$
5	$\left(65785\sqrt{112}x_{n+2} - 259\sqrt{112}x_{n+3}, 2741x_{2n+4} - 696203x_{2n+3}\right)$	$X^2 = 54Y - 11664$
6	$\left(22\sqrt{112}x_{n+2} - 518\sqrt{112}y_{n+1}, 5482y_{2n+2} - 224x_{2n+3}\right)$	$X^2 = 1143Y - 5225796$
7	$\left(5482\sqrt{112}x_{n+2} - 518\sqrt{112}y_{n+2}, 5482y_{2n+3} - 58016x_{2n+3}\right)$	$X^2 = 9Y - 324$
8	$\left(1392406\sqrt{112}x_{n+2} - 518\sqrt{112}y_{n+3}, 5482y_{2n+4} - 14735840x_{2n+3}\right)$	$X^2 = 1143Y - 5225796$
9	$\left(22\sqrt{112}x_{n+3} - 131570\sqrt{112}y_{n+1}, 1392406y_{2n+2} - 224x_{2n+4}\right)$	$X^2 = 290313Y - 337126551876$
10	$\left(5482\sqrt{112}x_{n+3} - 131570\sqrt{112}y_{n+2}, 1392406y_{2n+3} - 58016x_{2n+4}\right)$	$X^2 = 1143Y - 5225796$
11	$\left(1392406\sqrt{112}x_{n+3} - 131570\sqrt{112}y_{n+3}, 1392406y_{2n+4} - 14735840x_{2n+4}\right.$	$X^2 = 9Y - 324$
12	$\left(11y_{n+2} - 2741y_{n+1}, 259y_{2n+2} - y_{2n+3}\right)$	$X^2 = 6048Y - 1306368$

13	$\left(11y_{n+3}-696203y_{n+1}, 65785y_{2n+2}-y_{2n+4}\right)$	$X^2 = 1536192Y - 8428163788$
14	$\left(2741y_{n+3}-696203y_{n+2}, 65785y_{2n+3}-259y_{2n+4}\right)$	$X^2 = 6048Y - 1306368$

III. Consider $m = x_{n+1} + y_{n+1}, n = x_{n+1}$, observe that $m > n > 0.$ Treat m, n as the generators of the pythagorean triangle T (α, β, γ),

$\alpha = 2mn$, $\beta = m^2 - n^2$, $\gamma = m^2 + n^2$.

Then the following interesting relations are observed.

a) $\alpha + 55\gamma - 56\beta = -9$

b) $57\alpha - \gamma + 9 = \frac{224A}{P}$

c) $\frac{2A}{P} = x_{n+1}y_{n+1}$

d) $29\alpha - 28\beta + 27\gamma - \frac{112A}{P} = -9$

CONCLUSION

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive pell equation $y^2 = 112x^2 + 9$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of positive pell equations and determine their integer solutions along with suitable properties.

ACKNOWLEDGEMENT

The financial support from the UGC, New Delhi (F-MRP-5123/14(SERO/UGC) dated march 2014) for a part of this work is gratefully acknowledged.

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